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Localisation of electromagnetic waves in a randomly stratified medium

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Abstract. We study the way in which electromagnetic waves at arbitrary angles of incidence are localised in the direction normal to the strata of a randomly stratified medium. We point out differences from the case of normal incidence and, in particular, differences between the two polarisation modes which amount to a type of birefringence.

1. Introduction

There has been considerable interest in the localisation of classical waves [1-13] in random media. One reason for this is that classical waves obey linear equations of motion and should exhibit the consequences of localisation in a clearcut way. By contrast, the localisation of electrons in amorphous materials can be affected by electron-electron interactions [12].

In this paper we examine the problem of electromagnetic waves propagated in a randomly stratified medium. The case of waves incident normal to the strata has been considered by a number of authors [6-9]. Here we consider the more general case of a wave with an arbitrary angle of incidence. In contrast to the case of normal incidence the two polarisation modes behave in distinct, though similar, ways. They exhibit therefore a kind of birefringence. The equations for both modes are analogous to the Schrödinger equation for an electron in a one-dimensional random potential provided we allow for the possibility that it has a fluctuating effective mass [14].

2. Maxwell's equations in a stratified medium

We will assume that the randomness of the medium is represented by a position-dependent dielectric constant $\varepsilon = \varepsilon_0 f(\boldsymbol{r})$. Later we specialise to the case $f(\boldsymbol{r}) = f(z)$. The displacement vector is

$$\boldsymbol{D} = \varepsilon_0 f(\boldsymbol{r})\boldsymbol{E}. \tag{2.1}$$

In an appropriate gauge, the electric and magnetic fields are related to the vector potential $\boldsymbol{A}$. Thus

$$\boldsymbol{E} = -\dot{\boldsymbol{A}} \tag{2.2}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{2.3}$$

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In the absence of free charge and current density Maxwell's equations imply that

$$\nabla \cdot (f(\boldsymbol{r})\dot{\boldsymbol{A}}) = 0 \tag{2.4}$$

and

$$(1/c^2)f(\boldsymbol{r})\ddot{\boldsymbol{A}} + \nabla \times (\nabla \times \boldsymbol{A}) = 0 \tag{2.5}$$

where c is the velocity of light.

For a stratified medium $f(\boldsymbol{r}) = f(z)$. Let $\boldsymbol{n}$ be the unit vector in the z direction. The polarisation modes which can be propagated in the medium have the form

$$\boldsymbol{A} = \exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}_\perp \cdot \boldsymbol{r})\boldsymbol{\phi}(z) \tag{2.6}$$

where $\boldsymbol{n} \cdot \boldsymbol{k}_\perp = 0$. If we make the replacements

$$\frac{\partial}{\partial t} \to -\mathrm{i}\omega \tag{2.7}$$

$$\nabla \to \mathrm{i}\boldsymbol{k}_\perp + \boldsymbol{n}\frac{\partial}{\partial z} \tag{2.8}$$

then (2.4) and (2.5) become

$$\left(\mathrm{i}\boldsymbol{k}_\perp + \boldsymbol{n}\frac{\partial}{\partial z}\right) \cdot \boldsymbol{f}(\boldsymbol{\phi}) = 0 \tag{2.9}$$

and

$$-\frac{\omega^2}{c^2}f(z)\boldsymbol{\phi} + (\mathrm{i}\boldsymbol{k}_\perp + \boldsymbol{n}\partial/\partial z) \times [(\mathrm{i}\boldsymbol{k}_\perp + \boldsymbol{n}\partial/\partial z) \times \boldsymbol{\phi}] = 0. \tag{2.10}$$

For $\omega \neq 0$, (2.9) is implied by (2.10) and so is redundant.

The two polarisation modes have the forms:

(i) the orthogonal mode, for which

$$\boldsymbol{\phi} = \boldsymbol{e}\psi(z) \tag{2.11}$$

where $\boldsymbol{e} \cdot \boldsymbol{k}_\perp = \boldsymbol{e} \cdot \boldsymbol{n} = 0$;

(ii) the parallel mode, for which

$$\boldsymbol{\phi} = \alpha(z)\hat{\boldsymbol{k}}_\perp + \beta(z)\boldsymbol{n} \tag{2.12}$$

where $\hat{\boldsymbol{k}}_\perp$ is the unit vector parallel to $\boldsymbol{k}_\perp$.

Equation (2.10) implies the following equations of motion. For the orthogonal mode

$$\psi''(z) + \left(\frac{\omega^2}{c^2}f(z) - k_\perp^2\right)\psi(z) = 0. \tag{2.13}$$

For the parallel mode

$$\begin{pmatrix} \partial/\partial z \\ -\mathrm{i}k_\perp \end{pmatrix} \left(\frac{\partial}{\partial z}, -\mathrm{i}k_\perp\right) \begin{pmatrix} \alpha(z) \\ \beta(z) \end{pmatrix} + \frac{\omega^2}{c^2}f(z)\begin{pmatrix} \alpha(z) \\ \beta(z) \end{pmatrix} = 0 \tag{2.14}$$

where we have used an obvious matrix notation.

Clearly (2.13) for the transverse mode is exactly similar to the Schrödinger equation for an electron moving in a one-dimensional potential. Equation (2.14) for the parallel mode is a little complicated. The analogy among all the equations can be brought out by expressing them in first-order form.

If we set

$$\frac{\omega}{c}\phi = \psi' \tag{2.15}$$

then (2.13) is equivalent to

$$\begin{pmatrix}\psi'\\ \phi'\end{pmatrix} = \frac{\omega}{c}\begin{pmatrix}0 & 1\\ k_\perp^2/k^2 - f(z) & 0\end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix} \tag{2.16}$$

where $k = \omega/c$ is the vacuum wavenumber. If we introduce $\gamma(z)$ so that

$$\frac{\omega}{c}\gamma = \mathrm{i}k_\perp\beta - \alpha' \tag{2.17}$$

then (2.14) implies that

$$\begin{pmatrix}\gamma'\\ \alpha'\end{pmatrix} = \frac{\omega}{c}\begin{pmatrix}0 & f(z)\\ \dfrac{k_\perp^2}{k^2}\dfrac{1}{f(z)} - 1 & 0\end{pmatrix}\begin{pmatrix}\gamma\\ \alpha\end{pmatrix}. \tag{2.18}$$

The Schrödinger equation for the electron moving in a one-dimensional potential $V(z)$, with fluctuating effective mass $m(z)$, at energy E is

$$-\tfrac{1}{2}\hbar^2\left(\frac{1}{m(z)}\psi'\right)' + (V(z) - E)\psi(z) = 0. \tag{2.19}$$

Setting

$$\phi(z) = \frac{\hbar}{m(z)}\psi'(z) \tag{2.20}$$

we obtain the first-order equation

$$\begin{pmatrix}\psi'\\ \phi'\end{pmatrix} = \frac{1}{\hbar}\begin{pmatrix}0 & m(z)\\ -2(E - V(z)) & 0\end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix}. \tag{2.21}$$

Clearly all three problems have a similar form and this is identical to the form of the equations for normal incidence discussed in [6-9].

The analogy with the electron case is interesting. For both the parallel and orthogonal modes the vacuum wavenumber $k = \omega/c$ plays the role of $(\hbar)^{-1}$. Large frequency corresponds to $\hbar \to 0$ ('classical' limit) and low frequency to $\hbar \to \infty$ ('quantum' limit). For the orthogonal mode the quantity corresponding to the effective mass is constant, while that corresponding to the energy is

$$E_1 = \bar{f} - k_\perp^2/k^2 \tag{2.22}$$

where $\bar{f}$ is the mean value of f. For the parallel mode $f(z)$ plays the role of the effective mass and the energy parameter is

$$E_2 = 1 - (k_\perp^2/k^2)\overline{(f^{-1})}. \tag{2.23}$$

This analogy alerts us to the possibility that physical electromagnetic modes exist for which either or both of E_1 and E_2 are negative. Since we assume $f(z) \geqslant 1$, these modes have transverse wavenumbers satisfying

$$k_\perp > k \tag{2.24}$$

and therefore do not correspond to a real angle of incidence from the vacuum. Moreover, in the electron problem we have

$$E \geqslant \min(V(z)) \tag{2.25}$$

and in the electromagnetic case the corresponding condition

$$k_\perp^2/k^2 \leqslant \max(f(z)). \tag{2.26}$$

The physical picture then is that the waves with values of $k_\perp$ near its upper bound are constrained by internal reflection to travel along planes of highest dielectric constant, just as the most tightly bound electrons are trapped at the bottoms of the deepest wells. These deeply trapped modes are a new feature of the electromagnetic waves which is absent for the case of normal incidence ($k_\perp = 0$).

3. Characteristic probability distribution

The first-order equations of the previous section all have the form

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \lambda \begin{pmatrix} 0 & \rho(z) \\ -\sigma(z) & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.1}$$

where $\rho(z)$ is a positive quantity. Following Halpern [14] we introduce a variable

$$\zeta = v/u \tag{3.2}$$

which then satisfies the differential equation

$$\zeta' = -\lambda(\sigma(z) + \rho(z)\zeta^2). \tag{3.3}$$

As in [6-9] the statistical features of the model are introduced by allowing σ and ρ to become functions of an n-dimensional vector-valued stochastic variable $\xi(z)$. Equation (3.3) then becomes a stochastic differential equation for ζ.

The induced probability distribution for ξ, $P(\xi)$, satisfies [6]

$$Q^*\bar{P}(\xi) = 0 \tag{3.4}$$

where Q^* is the operator adjoint to the operator Q which is defined through the stochastic process $\xi(z)$ by the formula

$$Qf(\xi_0) = \lim_{h\to 0} \frac{1}{h} E[f(\xi(z+h)) - f(\xi_0)|\xi(z) = \xi_0]. \tag{3.5}$$

The joint probability distribution for ξ and ζ then satisfies

$$Q^*P(\xi, \zeta) + \lambda \frac{\partial}{\partial \zeta}(\sigma(\xi) + \rho(\xi)\zeta^2)P(\xi, \zeta) = 0. \tag{3.6}$$

Integrating over ξ we find

$$\frac{\partial}{\partial \zeta} \int \mathrm{d}\xi (\sigma(\xi) + \rho(\xi)\zeta^2) P(\xi, \zeta) = 0. \tag{3.7}$$

This approach allows us to discuss a wide class of statistical models.

The physical properties of the system can be computed from the probability distribution $P(\xi, \zeta)$. One particularly important property is the extent to which the localised mode functions are spread out in space. This can be specified by means of a localisation length l defined in the following way. First define a measure of 'size', r, for the mode function by the equation

$$r^2 = pu^2 + qv^2 \tag{3.8}$$

where p and q are positive quantities. Next, we compute the mean value γ of the logarithmic rate of change of r:

$$\gamma = \overline{(\ln r)'}. \tag{3.9}$$

The average shape of a mode will then roughly behave like $\exp(\pm\gamma z)$. We now identify the localisation length l with γ^{-1}. The precise value of l depends on the particular choice of p and q and is, to some extent, arbitrary. A fairly natural choice which we will adopt is

$$p = |\bar{\sigma}| \qquad q = \bar{\rho}. \tag{3.10}$$

We then find

$$l^{-1} = \lambda \int d\xi d\zeta P(\xi, \zeta) \frac{(|\bar{\sigma}|\rho(\xi) - \bar{\rho}\sigma(\xi))\zeta}{|\bar{\sigma}| + \bar{\rho}\zeta^2}. \tag{3.11}$$

In the limit of large λ, (3.6) is dominated by the 'convection' term and becomes

$$\frac{\partial}{\partial\zeta}(\sigma(\xi) + \rho(\xi)\zeta^2)P(\xi, \zeta) = 0. \tag{3.12}$$

It is clear that in this limit we must distinguish between two cases:

(i) $\sigma(\xi) > 0$ for all ξ,

(ii) $\sigma(\xi)$ fluctuating in sign.

In case (i), the positive case, (3.12) has a solution

$$P(\xi, \zeta) = \frac{1}{\pi}\bar{P}(\xi)\frac{[\sigma(\xi)\rho(\xi)]^{1/2}}{\sigma(\xi) + \rho(\xi)\zeta^2} \tag{3.13}$$

where we have imposed the normalisation condition

$$\int d\zeta P(\xi, \zeta) = \bar{P}(\xi). \tag{3.14}$$

In case (ii), the fluctuating case, we must modify (3.13) in regions where $\sigma(\xi) < 0$, since otherwise the probability distribution would acquire poles in integration range at points $\zeta = \pm\alpha(\xi)$, where $\alpha(\xi) = [-\sigma(\xi)/\rho(\xi)]^{1/2}$. This is unacceptable. Nevertheless there is a distribution solution of (3.13) when $\sigma(\xi) < 0$, namely a combination of δ functions with support at the zeros of the convective flow, $\zeta = \pm\alpha(\xi)$. The zero at $\zeta = -\alpha(\xi)$ is unstable since the flow conveys the probability distribution away from this point. The zero at $\zeta = \alpha(\xi)$ is stable since the flow sweeps any distribution into this point. Retaining only the stable δ function we find the modified form of (3.13) is

$$P(\xi, \zeta) = \bar{P}(\xi)\left(\theta(\sigma(\xi))\frac{1}{\pi}\frac{[\sigma(\xi)\rho(\xi)]^{1/2}}{\sigma(\xi) + \rho(\xi)\zeta^2} + \theta(-\sigma(\xi))\delta(\zeta - \alpha(\xi))\right) \tag{3.15}$$

where $\theta(x)$ is the Heaviside step function.

From this result we can compute the localisation length in the limit of large λ. We then have

$$l^{-1} = \lambda \int d\xi\, \theta(-\sigma(\xi))\bar{P}(\xi)[-\sigma(\xi)/\rho(\xi)]^{1/2}. \tag{3.16}$$

We then find contrasting behaviour in the positive and fluctuating cases. In the positive case there is no contribution of $O(\lambda)$ to l^{-1}. It was pointed out in [6] that in this case the first non-vanishing contributions are of lower order in λ and are model dependent. We will return to this point in the next section. In the fluctuating case, the result is model independent and is dominated by the regions of negative $\sigma(\xi)$.

When λ is very small, (3.6) is dominated by the diffusion term and becomes

$$Q^* P(\xi, \zeta) = 0 \tag{3.17}$$

with a solution

$$P(\xi, \zeta) = \bar{P}(\xi)\phi(\zeta). \tag{3.18}$$

From (3.7) it follows that

$$\frac{\partial}{\partial \zeta}(\bar{\sigma} + \bar{\rho}\zeta^2)\phi(\zeta) = 0. \tag{3.19}$$

Again we have two cases:
(i) $\bar{\sigma} > 0$,
(ii) $\bar{\sigma} < 0$.
For case (i) we find

$$P(\xi, \zeta) = \frac{1}{\pi} \bar{P}(\xi) \frac{(\bar{\sigma}\bar{\rho})^{1/2}}{\bar{\sigma} + \bar{\rho}\zeta^2}. \tag{3.20}$$

For case (ii)

$$P(\xi, \zeta) = \bar{P}(\xi)\delta(\zeta - \bar{\alpha}) \tag{3.21}$$

where $\bar{\alpha} = (-\bar{\sigma}/\bar{\rho})^{1/2}$. When $\bar{\sigma} > 0$ we see again that the leading term $O(\lambda)$, which contributes to l^{-1}, vanishes. The first non-vanishing contribution is $O(\lambda^2)$.

When $\bar{\sigma} < 0$, however, we do obtain such a contribution, namely

$$l^{-1} = \lambda(-\bar{\sigma}\bar{\rho})^{1/2}. \tag{3.22}$$

4. Slab models

In order to illustrate a higher-order calculation we consider a model discussed in [6], which comprises slabs of material within which the random variable ξ attains a constant value which is distributed according to the distribution $\bar{P}(\xi)$, independently for each slab. The thickness of a slab is also a random variable distributed according to a Poisson distribution. If the mean thickness of the slabs is a, then from (3.5) we find

$$Qf(\xi_0) = \frac{1}{a}\left(\int d\xi \bar{P}(\xi) f(\xi) - f(\xi_0)\right) \tag{4.1}$$

and hence

$$Q^* f(\xi_0) = \frac{1}{a}\left(\bar{P}(\xi_0) \int d\xi f(\xi) - f(\xi_0)\right). \tag{4.2}$$

For the case of positive $\sigma(\xi)$, we can obtain higher-order corrections to (3.13) by setting

$$P(\xi, \zeta) = \sum_{n=0}^{\infty} \lambda^{-n} P_n(\xi, \zeta) \tag{4.3}$$

where $P_0(\xi, \zeta)$ is given by (3.13). We find from (3.6) for $n \leqslant 1$

$$Q^* P_{n-1}(\xi, \zeta) + \frac{\partial}{\partial \zeta}(\sigma(\xi) + \rho(\xi)\zeta^2) P_n(\xi, \zeta) = 0. \quad (4.4)$$

Taking into account the correct normalisation condition for $P(\xi, \zeta)$ we find from (4.4), for $n = 1$, that

$$P_1(\xi, \zeta) = -\frac{1}{\sigma(\xi) + \rho(\xi)\zeta^2} \int_{-\infty}^{\zeta} \mathrm{d}\zeta' Q^* P_0(\xi, \zeta'). \quad (4.5)$$

The contribution to the inverse correlation length is, of course,

$$l^{-1} = -\int_{-\infty}^{\infty} \mathrm{d}\zeta \int \mathrm{d}\zeta \frac{(\bar{\sigma}\rho(\xi) - \bar{\rho}\sigma(\xi))\zeta}{\bar{\sigma} + \bar{\rho}\zeta^2} P_1(\xi, \zeta). \quad (4.6)$$

Using (4.6) and (4.2) we find for the slab models described above that

$$l^{-1} = \frac{1}{2a} \int \mathrm{d}\xi \int \mathrm{d}\xi' \bar{P}(\xi) \bar{P}(\xi') \ln[\tfrac{1}{4}(1 + X)(1 + X^{-1})] \quad (4.7)$$

where

$$X = \left(\frac{\rho(\xi)\sigma(\xi')}{\sigma(\xi)\rho(\xi')}\right)^{1/2}. \quad (4.8)$$

When $\sigma(\xi)$ can change in sign it is not so straightforward to calculate higher-order corrections. In the appendix we analyse a simple two-component slab model to show how the δ-function distribution is approached in the limit of large λ. These results are consistent with those of [6].

To study the limit of small λ we expand $P(\xi, \zeta)$ thus:

$$P(\xi, \zeta) = \sum_0^{\infty} \lambda^n p_n(\xi, \zeta). \quad (4.9)$$

Of course $p_0(\xi, \zeta)$ is given by (3.20) when $\bar{\sigma} > 0$. From (3.6) and (3.7) we find, for $n \leqslant 1$,

$$Q^* p_n(\xi, \zeta) + \frac{\partial}{\partial \zeta}(\sigma(\xi) + \rho(\xi)\zeta^2) p_{n-1}(\xi, \zeta) = 0 \quad (4.10)$$

and

$$\frac{\partial}{\partial \zeta} \int \mathrm{d}\xi (\sigma(\xi) + \rho(\xi)\zeta^2) p_n(\xi, \zeta) = 0. \quad (4.11)$$

It follows from (4.10), in the slab model, that

$$p_1(\xi, \zeta) = \bar{P}(\xi)\phi_1(\zeta) + a\frac{\partial}{\partial \zeta}(\sigma(\xi) + \rho(\xi)\zeta^2) p_0(\xi, \zeta) \quad (4.12)$$

where

$$\int \mathrm{d}\zeta \phi_1(\zeta) = 0. \quad (4.13)$$

From (4.11) we have

$$\frac{\partial}{\partial \zeta}(\bar{\sigma} + \bar{\rho}\zeta^2)\phi_1(\zeta) + a\frac{\partial}{\partial \zeta} \int \mathrm{d}\xi \bar{P}(\xi)(\sigma(\xi) + \rho(\xi)\zeta^2) \frac{\partial}{\partial \zeta}(\sigma(\xi) + \rho(\xi)\zeta^2)\phi_0(\zeta) = 0. \quad (4.14)$$

Taking account of (4.13), and using the appropriate form for $\phi_0(\zeta)$, we find

$$\phi_1(\zeta) = -\frac{a}{2\pi}(\bar{\sigma}\bar{\rho})^{1/2}\int d\xi \bar{P}(\xi)\frac{\partial}{\partial\zeta}\left(\frac{\sigma+\rho\zeta^2}{\bar{\sigma}+\bar{\rho}\zeta^2}\right)^2. \tag{4.15}$$

The result for the inverse localisation length is finally

$$l^{-1} = \frac{1}{4}\frac{\lambda^2 a}{\bar{\sigma}\bar{\rho}}(\bar{\sigma}^2\overline{\delta\rho^2}+\bar{\rho}^2\overline{\delta\sigma^2}-2\bar{\sigma}\bar{\rho}\overline{\delta\rho\delta\sigma}). \tag{4.16}$$

Again this is consistent with results quoted in [6].

When $\bar{\sigma} < 0$ the corrections are more difficult to compute.

5. Interpretation of the results

The results of the previous sections may be applied to the propagation of electromagnetic waves by identifying λ with the vacuum wavenumber $k = \omega/c$. Large and small values of λ therefore correspond to high and low frequencies, the limits being taken for fixed values of the ratio $k_\perp/k$.

For the orthogonal mode we have

$$\rho(\xi) = 1 \tag{5.1}$$

$$\sigma(\xi) = f(\xi) - k_\perp^2/k^2. \tag{5.2}$$

The case $\sigma(\xi) > 0$ for all ζ corresponds to the regime

$$k_\perp^2/k^2 < \min f(\xi) \tag{5.3}$$

which includes the range $k_\perp^2/k^2 \leqslant 1$ for which the vacuum incidence angle θ, given by

$$\sin\theta = k_\perp/k \tag{5.4}$$

is real. For values of $k_\perp/k$ satisfying the inequality in (5.3), our analysis confirms the results of [6] and shows that, in the slab model, the localisation length $l_\perp$ has a finite limit at high frequencies given by

$$l_\perp^{-1} = \frac{1}{2a}\int d\xi d\xi' \bar{P}(\xi)\bar{P}(\xi')\ln\left\{\frac{1}{4}\left[1+\left(\frac{f(\xi)-k_\perp^2/k^2}{f(\xi')-k_\perp^2/k^2}\right)^{1/2}\right]\left[1+\left(\frac{f(\xi')-k_\perp^2/k^2}{f(\xi)-k_\perp^2/k^2}\right)^{1/2}\right]\right\}. \tag{5.5}$$

In the case of a weakly random medium, for which

$$\overline{\delta f^2} \ll (\bar{f} - k_\perp^2/k^2)^2 \tag{5.6}$$

we find

$$l_\perp^{-1} = \frac{1}{16}\frac{1}{a}\frac{\overline{\delta f^2}}{(\bar{f}-k_\perp^2/k^2)^2}. \tag{5.7}$$

For the parallel mode we have

$$\rho(\xi) = f(\xi) \tag{5.8}$$

$$\sigma(\xi) = 1 - k_\perp^2/k^2\frac{1}{f(\xi)}. \tag{5.9}$$

The slab model result in the high-frequency limit becomes

$$l_{\|}^{-1}=\frac{1}{2a}\int \mathrm{d}\xi\,\mathrm{d}\xi'\bar{P}(\xi)\bar{P}(\xi')\ln\left\{\frac{1}{4}\left[1+\frac{f(\xi)}{f(\xi')}\left(\frac{f(\xi')-k_\perp^2/k^2}{f(\xi)-k_\perp^2/k^2}\right)^{1/2}\right]\right.$$
$$\left.\times\left[1+\frac{f(\xi')}{f(\xi)}\left(\frac{f(\xi)-k_\perp^2/k^2}{f(\xi')-k_\perp^2/k^2}\right)^{1/2}\right]\right\}. \tag{5.10}$$

Note that

$$l_\perp \neq l_\|. \tag{5.11}$$

This is an example of birefringent 'localisation'. In the weakly random case we have

$$l_\|^{-1}=\frac{1}{16a}\frac{\overline{\delta f^2}}{(\bar{f}-k_\perp^2/k^2)^2}\left(1+\frac{2k_\perp^2}{\bar{f}k^2}\right)^2. \tag{5.12}$$

We see that for small $\overline{\delta f^2}$ both $l_\perp$ and $l_\|$ are much greater than a, the mean slab thickness, and both are sensitive to the value of $k_\perp/k$, decreasing as $k_\perp/k$ is increased from zero. The rate of decrease is however different for the two cases since

$$\frac{l_\perp}{l_\|}=\left(1+\frac{2}{\bar{f}}\frac{k_\perp^2}{k^2}\right)^2 \tag{5.13}$$

and this increases with $k_\perp/k$. It follows that the orthogonal mode is less strongly localised than the parallel mode when the angle of incidence is increased from zero. Although these results have been obtained in a slab model they presumably give an indication of the qualitative behaviour of other models.

When $k_\perp^2/k^2$ is so large that we enter the regime where $\sigma(\xi)$ fluctuates in sign we find an altered model-independent behaviour for the localisation length at high frequencies. By referring to (5.1), (5.2), (5.8), (5.9) and (3.16) we see that both the orthogonal and parallel modes have the same limiting behaviour, namely

$$l_\perp = l_\| \approx \int \mathrm{d}\xi\bar{P}(\xi)\theta(k_\perp^2/k^2-f(\xi))(k_\perp^2-f(\xi)k^2)^{1/2}. \tag{5.14}$$

This result has a natural interpretation. We can visualise a mode passing through regions for which σ is positive and those for which it is negative. In the former the wave will be oscillatory with essentially a zero value for the logarithmic derivative, γ, locally. In the latter the wave is naturally damped with a value for γ given by

$$\gamma=(k_\perp^2-f(\xi)k^2)^{1/2}. \tag{5.15}$$

It is then plausible that the mean value of γ which yields l^{-1} is given by an average over the regions where γ is non-vanishing. This is just what is provided by (5.14). With this interpretation it is not surprising that the two modes behave in the same way.

We see from (5.1) and (5.2) that, for the orthogonal mode,

$$\begin{aligned}&\bar{\rho}=1\\ &\bar{\sigma}=\bar{f}-k_\perp^2/k^2\\ &\overline{\delta\rho^2}=\overline{\delta\rho\delta\sigma}=0\\ &\overline{\delta\sigma^2}=\overline{\delta f^2}.\end{aligned} \tag{5.16}$$

It follows then from (4.16) that, in the low-frequency limit,

$$l_\perp^{-1}=\frac{1}{4}a\frac{\omega^2}{c^2}\frac{\overline{\delta f^2}}{(\bar{f}-k_\perp^2/k^2)}. \tag{5.17}$$

Similarly from (5.8) and (5.9) we see for the parallel mode that

$$\begin{aligned}
&\bar{\rho}=\bar{f}\\
&\overline{\delta\rho^2}=\overline{\delta f^2}\\
&\bar{\sigma}=1-\frac{k_\perp^2}{k^2}\overline{(f^{-1})}\\
&\overline{\delta\sigma^2}=(k_\perp^2/k^2)^2\overline{(\delta f^{-1})^2}\\
&\overline{\delta\sigma\delta\rho}=\frac{k_\perp^2}{k^2}(\bar{f}\overline{f^{-1}}-1).
\end{aligned} \tag{5.18}$$

For weak disorder

$$\begin{aligned}
&\overline{(f^{-1})}=\frac{1}{\bar{f}}+\frac{\overline{\delta f^2}}{\bar{f}^3}\\
&\overline{(\delta f^{-1})^2}=\frac{\overline{\delta f^2}}{\bar{f}^4}\\
&\overline{\delta\sigma\delta\rho}=\frac{k_\perp^2}{k^2}\frac{\overline{\delta f^2}}{\bar{f}^2}.
\end{aligned} \tag{5.19}$$

In that case we have from (4.16)

$$l_\parallel^{+1}\simeq\tfrac{1}{4}a\frac{\omega^2}{c^2}\frac{\overline{\delta f^2}}{(\bar{f}-k_\perp^2/k^2)}(1-2k_\perp^2/\bar{f}k^2)^2. \tag{5.20}$$

Once again we see that there is a difference between the localisation lengths of the two modes. In the low-frequency limit we find

$$l_\perp/l_\parallel\simeq(1-2k_\perp^2/\bar{f}k^2). \tag{5.21}$$

That is, the orthogonal mode is more strongly localised than the parallel mode, reversing the situation which holds in the high-frequency limit. Again the effect is stronger the greater the angle of incidence.

In the regime where $\bar{\sigma}<0$ we see from (3.22) that

$$l_\perp^{-1}=(k_\perp^2-\bar{f}k^2)^{1/2} \tag{5.22}$$

and

$$l_\parallel^{-1}=[\bar{f}\overline{(f^{-1})}k_\perp^2-\bar{f}k^2]^{1/2}. \tag{5.23}$$

For the orthogonal mode the medium behaves as if it had an effective dielectric constant $\varepsilon_0\bar{f}$. This is not quite true for the parallel mode because of the difference between $(\bar{f})^{-1}$ and $\overline{(f^{-1})}$. For weak disorder

$$\frac{l_\perp}{l_\parallel}\simeq 1+\frac{1}{2}\frac{k_\perp^2}{k_\perp^2-\bar{f}k^2}\frac{\overline{\delta f^2}}{\bar{f}^2}.$$

6. Conclusions

We have examined the propagation of electromagnetic waves in randomly stratified media. Both polarisation modes satisfy similar equations and exhibit the localisation effect in similar ways. There are differences between the modes in their detailed behaviour, however [15]. For real vacuum incidence angles our analysis of slab models suggests that at high frequencies the orthogonal mode is less localised than the parallel mode, while at low frequencies the reverse is true. Both effects increase with increasing angle of incidence. When the transverse wavenumber if sufficiently large, the nature of the waves changes. They become trapped waves which exist due to internal reflection. For such waves the high-frequency behaviour of the localisation length is model independent and identical for both polarisations. At low frequencies, however, the distinction between the modes again appears, the parallel mode being more strongly localised than the orthogonal model. Further work on other models is desirable but requires an extensive numerical investigation.

The reflection of electromagnetic radiation at the surface of a randomly stratified medium is a problem of considerable interest with potential applications to microwave investigations of the Earth's surface. Some of the properties of such reflected waves are determined by the probability distributions we have studied here. However, other problems, such as the depolarisation of an incident plane-polarised wave, require an investigation of a joint probability distribution of the ζ variables associated with both polarisation modes. We hope to examine these questions in future work.

Finally it would be interesting to investigate the trapped waves which exist for large values of the transverse wavevector. One possible method of stimulation is by the passage of a fast charge particle through the medium emitting localised Cerenkov radiation. The theoretical analysis of such a process requires an understanding of the density of states for the trapped waves.

Appendix

In this appendix we examine a simple two-component slab model in order to understand the difference between the regimes in which $\sigma(\xi)$ is strictly positive and in which it fluctuates in sign. We let ξ be a random variable taking the values ± 1 with equal probability and set

$$\rho(\pm 1) = \rho_{\pm} \tag{A1}$$

$$\sigma(\pm) = \sigma_{\pm}. \tag{A2}$$

We assume $\sigma_+ > \sigma_-$ so the fluctuating regime corresponds to the case where $\sigma_- < 0$. We then have

$$\bar{P}(\xi) = \tfrac{1}{2}\delta(\xi - 1) + \tfrac{1}{2}\delta(\xi + 1). \tag{A3}$$

The characteristic probability distribution for ξ and ζ has the form

$$P(\xi, \zeta) = f(\zeta)(\xi - 1) + g(\zeta)\delta(\xi + 1). \tag{A4}$$

If we form the two-component vector

$$x(\zeta) = \begin{pmatrix} f(\zeta) \\ g(\xi) \end{pmatrix} \tag{A5}$$

then it is easy to deduce from (3.6) and (4.2) that

$$-\frac{1}{a}vv^{\mathrm{T}}x(\zeta)+\lambda\frac{\partial}{\partial\zeta}M(\zeta)x(\zeta)=0 \tag{A6}$$

where

$$M(\zeta)=\begin{pmatrix}\sigma_{+}+\rho_{+}\zeta^{2} & 0\\ 0 & \sigma_{-}+\rho_{-}\zeta^{2}\end{pmatrix} \tag{A7}$$

and

$$v=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}. \tag{A8}$$

We also introduce the orthogonal vector:

$$w=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}. \tag{A9}$$

If we denote the inverse of $M(\zeta)$ by $\bar{M}(\zeta)$, then

$$\bar{M}(\zeta)=\begin{pmatrix}\dfrac{1}{\sigma_{+}+\rho_{+}\zeta^{2}} & 0\\ 0 & \dfrac{1}{\sigma_{-}+\rho_{-}\zeta^{2}}\end{pmatrix}. \tag{A10}$$

Introducing $y(\zeta)$ where

$$y(\zeta)=M(\zeta)x(\zeta) \tag{A11}$$

we find from (A6) that

$$\frac{\partial}{\partial\zeta}y(\zeta)=\frac{1}{\lambda a}vv^{\mathrm{T}}\bar{M}(\zeta)y(\zeta). \tag{A12}$$

A solution to this equation can be obtained in the following way. First, note that

$$\frac{\partial}{\partial\zeta}w^{\mathrm{T}}y=0 \tag{A13}$$

so that

$$w^{\mathrm{T}}y=N \tag{A14}$$

where N is a constant of integration. Then, using the identity

$$vv^{\mathrm{T}}+ww^{\mathrm{T}}=1 \tag{A15}$$

we see that

$$\frac{\partial}{\partial\zeta}(v^{\mathrm{T}}y)=\frac{1}{\lambda a}\{h(\zeta)(v^{\mathrm{T}}y)+Nv^{\mathrm{T}}\bar{M}(\zeta)w\} \tag{A16}$$

where

$$h(\zeta)=v^{\mathrm{T}}\bar{M}(\zeta)v. \tag{A17}$$

Now introduce $F(\zeta)$ such that

$$\frac{\partial}{\partial\zeta} F(\zeta) = \frac{1}{\lambda a} h(\zeta) F(\zeta). \tag{A18}$$

We then find

$$v^{\mathrm{T}} y = F(\zeta)\left(K + \frac{N}{\lambda a} \int_{\zeta_0}^{\zeta} \mathrm{d}\zeta' (v^{\mathrm{T}} \bar{M}(\zeta') w) F^{-1}(\zeta') \right) \tag{A19}$$

where K is a second constant of integration.

The difference between the two regimes, $\sigma_- > 0$ and $\sigma_- < 0$, is reflected in a change of character in the nature of $F(\zeta)$, the solution of (A18). When $\sigma_- > 0$ we can choose

$$F(\zeta) = \exp\left[\frac{1}{2\lambda a}\left(\frac{\chi_+(\zeta)}{(\rho_+\sigma_+)^{1/2}} + \frac{\chi_-(\zeta)}{(\rho_-\sigma_-)^{1/2}}\right)\right] \tag{A20}$$

where $\chi_\pm(\zeta)$ are defined through the relation

$$\zeta = \left(\frac{\sigma_\pm}{\rho_\pm}\right)^{1/2} \tan \chi_\pm(\zeta). \tag{A21}$$

It is then natural to choose $\zeta_0 = -\infty$; K and N are then fixed by the requirements that

$$\int_{-\infty}^{\infty} \mathrm{d}\zeta v^{\mathrm{T}} x(\zeta) = 0 \tag{A22}$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}\zeta w^{\mathrm{T}} x(\zeta) = 1/\sqrt{2}. \tag{A23}$$

Equation (A22) is equivalent to the condition that $(v^{\mathrm{T}} y(\zeta))$ has a common value in the two limits $\zeta \to \pm\infty$. By expanding in powers of $(\lambda a)^{-1}$ we can recover the large-λ expansion discussed in § 4.

When $\sigma_- < 0$, $\bar{M}(\zeta)$ has poles at the points

$$\zeta = \pm\left(\frac{-\sigma_-}{\rho_-}\right)^{1/2} = \pm\alpha. \tag{A24}$$

The presence of these poles on the real axis changes the character of the probability distribution $x(\zeta)$.

A useful way of investigating the nature of this change is to express the solution of (A12) as a power series thus:

$$y(\zeta) = \sum_{n=0}^{\infty} (\zeta - \alpha)^{n+\eta} y_n. \tag{A25}$$

On substituting into (A12) and equating the coefficients of $(\zeta - \alpha)^{n+\eta}$ to zero we obtain for the case $n = 0$

$$L y_0 = 0 \tag{A26}$$

where

$$L = \begin{pmatrix} \eta & \varepsilon \\ 0 & \eta - \varepsilon \end{pmatrix} \tag{A27}$$

with

$$\varepsilon = \frac{1}{4\lambda a(-\sigma_-\rho_-)^{1/2}}. \tag{A28}$$

Since we assume y_0 is not the null vector, we require L to be a singular matrix which yields the incidial equation

$$\eta(\eta-\varepsilon)=0. \tag{A29}$$

The solution corresponding to $\eta=0$ yields a dependence for $x(\zeta)$ which is smooth at $\zeta=\alpha$. The other solution of (A29) implies that

$$x(\zeta)\sim(\zeta-\alpha)^{-1+\varepsilon} \tag{A30}$$

when $\zeta\approx\alpha$. Although an unbounded singularity when ε is sufficiently small, it is still integrable and so cannot be ruled out as part of a probability distribution.

When we examine the solutions of (A12) in the neighbourhood of $\zeta=-\alpha$, we obtain an indicial equation

$$\eta(\eta+\varepsilon)=0. \tag{A31}$$

Again the solution $\eta=0$ yields a smooth function $x(\zeta)$ at $\zeta=-\alpha$. However the solution $\eta=-\varepsilon$ yields a dependence for $x(\zeta)$ of the form

$$x(\zeta)\sim(\zeta+\alpha)^{-1-\varepsilon} \tag{A32}$$

which can never be an integrable singularity. It follows that the correct solution of (A12) is that corresponding to the index $\eta=0$ at the point $\zeta=-\alpha$. Such a solution will have the integrable singularity indicated in (A30) as part of its structure.

This analysis is confirmed by examining the new structure of $F(\zeta)$ when $\sigma_-<0$. It is easily checked that a solution of (A18), for the range $-\alpha<\zeta<\alpha$, is

$$F(\zeta)=\left(\frac{\alpha-\zeta}{\alpha+\zeta}\right)^{\varepsilon}\exp\left(\frac{1}{2\lambda a}\frac{\chi_+}{(\sigma_+\rho_+)^{1/2}}\right). \tag{A33}$$

The solution which is smooth at $\zeta=-\alpha$ is obtained by setting $\zeta_0=-\alpha$ and $K=0$ in (A18). The result is

$$\begin{aligned} v^{\mathrm{T}}y(\zeta)=\frac{N}{2\lambda a}\left(\frac{\alpha-\zeta}{\alpha+\zeta}\right)^{\varepsilon}\exp\left(\frac{\chi_+(\zeta)}{2\lambda a(\rho_+\sigma_+)^{1/2}}\right)\int_{-\alpha}^{\zeta}\mathrm{d}\zeta'\left(\frac{1}{\rho_+\zeta'^2+\sigma_+}-\frac{1}{\rho_-\zeta'^2+\sigma_-}\right)\\ \times\left(\frac{\alpha+\zeta'}{\alpha-\zeta'}\right)^{\varepsilon}\exp\left(\frac{-\chi_+(\zeta')}{2\lambda a(\rho_+\sigma_+)^{1/2}}\right). \end{aligned} \tag{A34}$$

For $\zeta<-\alpha$ we analytically continue this result. Note that although, in (A34), both the integral and the factor

$$\left(\frac{\alpha-\zeta}{\alpha+\zeta}\right)^{\varepsilon}$$

have singularities at $\zeta=-\alpha$, these cancel in the product. As $\zeta\to\alpha$ the expression in (A34) shows a singularity structure of the type predicted by the power series analysis.

Returning to (A26) and (A27) we see that the choice $\eta = \varepsilon$ implies that, up to a normalisation,

$$y_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{A35}$$

This implies that, near $\zeta = \alpha$,

$$x(\zeta) = \begin{pmatrix} A|\zeta - \alpha|^{\varepsilon} \\ B|\zeta - \alpha|^{-1+\varepsilon} \end{pmatrix} + \text{finite}. \tag{A36}$$

We can see from this equation that, as $\lambda \to \infty$ and $\varepsilon \to 0$, the lower component of $x(\zeta)$ becomes more and more singular. To preserve the normalisation of $x(\zeta)$ it is necessary that $B \propto \varepsilon$. In the limit $\varepsilon \to 0$

$$\varepsilon|\zeta - \alpha|^{-1+\varepsilon} \sim \delta(\zeta - \alpha). \tag{A37}$$

If we accept this behaviour for $g(\zeta)$ we then deduce directly from (A6) that, as $\lambda \to \infty$,

$$x(\zeta) \to \frac{1}{2} \begin{pmatrix} \dfrac{(\rho_+ \sigma_+)^{1/2}}{\rho_+ \zeta^2 + \sigma_+} \\ \delta(\zeta - \alpha) \end{pmatrix} \tag{A38}$$

the relative normalisation being fixed by (A22). Equation (A38) is the analogue for this two-component model of (4.6). However, we can see clearly in this simple model the nature of the non-analytic behaviour at $\zeta = \alpha$ which gives rise to the above result for large λ.

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